A survey of new quantum "$az+b$" groups

Piotr Mikołaj Sołtan

Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw
piotr.soltan@fuw.edu.pl

Abstract

We present C*-algebraic quantum deformations of the "$az+b$" group for new values of the deformation parameter.

1 Introduction

Quantum "$az+b$" groups have been considered by several authors ([17], [10], [4]). These quantum groups are labeled by a complex parameter q called the deformation parameter. In the known approaches it was either a real number between 0 and 1 or a root of unity of the form $e^{\frac{2\pi i}{N}}$ with N an even integer greater or equal to 6.

In this paper we present the results of the authors thesis [8] in which a deformation of the "$az+b$" group was constructed with a complex deformation parameter of modulus strictly smaller than 1. The values of the parameter are restricted to a certain subset of the unit disk (cf. Subsection 2.1). It turns out that these values are the only ones for which the construction can be carried out within the scheme adopted in [17] and [8]. Moreover many properties of the associated special functions (cf. Subsection 2.2) are equivalent to the conditions imposed on the values of q.

We have decided to omit the proofs of most statements of the paper. Many of them are the same for quantum "$az+b$" groups regardless of the value of the deformation parameter. The analysis of Weyl commutation relations is also similar for all values of the parameter and is based on the results for the Zakrzewski commutation relations from [16]. The specific problems related to the deformation presented in this paper are mainly of computational nature and are all addressed in [8].

1.1 Description of the paper

In the next two subsections we briefly review the known quantum versions of the "$az+b$" group. We begin with the Hopf $*$-algebra level and then describe the C*-algebraic versions of S.L. Woronowicz.

Section 2 is devoted to the presentation of necessary instruments for the construction of our quantum group. In Section 3 we construct the main object in the construction of the quantum group – the multiplicative unitary operator, and examine its properties. Section 4 presents the construction of the quantum group as well as some additional results. In this section the connection with the algebraic situation of subsection 1.2 is made. In the appendix we present figures showing the set of allowable values of q and the group Γ_q (cf. Subsection 2.1).

1.2 Hopf $*$-algebra level

Let q be a non zero complex number and let $\mathscr{A}$ be a unital $*$-algebra generated by three normal elements a, a^{-1} and b subject to relations:

$$\begin{gathered} a^{-1}a = aa^{-1} = I, \\ ab = q^2 ba, \quad ab^* = b^* a. \end{gathered}$$

We endow $\mathscr{A}$ with the structure of a Hopf $*$-algebra putting

$$\begin{aligned} \delta(a) &= a \otimes a, & \epsilon(a) &= 1, & \kappa(a) &= a^{-1}, \\ \delta(b) &= a \otimes b + b \otimes I, & \epsilon(b) &= 0, & ,\kappa(b) &= -a^{-1}b. \end{aligned}$$

This Hopf $*$-algebra has some interesting properties. For example its antipode κ can be written as

$$\kappa = R \circ \tau_{i/2}, \tag{1}$$

where R is a $*$-antiautomorphism of $\mathscr{A}$ (called the *unitary antipode*) and $\tau_{i/2}$ is a holomorphic extension of a one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathscr{A}$ called the *scaling group* (cf. [10, Prop. 2.4] and [8, Prop. 4.1] for details). We refer to the decomposition (1) as the *polar decomposition* of κ.

1.3 Examples of S.L. Woronowicz

In [17] S.L. Woronowicz constructed quantum deformations of the "$az + b$" group on the level of C*-algebras. The algebraic relations of the Hopf $*$-algebra approach had to be supplemented by *spectral conditions.* For $q = e^{\frac{2\pi i}{N}}$ (with N an even integer greater or equal to 6) the conditions are that spectra of the generators a and b be contained in the closure of the set

$$\bigcup_{k=1}^{N-1} q^k \mathbb{R}_+ \subset \mathbb{C}.$$

It is remarkable that this condition can be put in the algebraic framework. It is equivalent to demand that $a^{\frac{N}{2}}$ and $b^{\frac{N}{2}}$ be selfadjoint. The quotient of the Hopf $*$-algebra $\mathscr{A}$ by the ideal generated by $a^{\frac{N}{2}} - \left(a^{\frac{N}{2}}\right)^*$ and $b^{\frac{N}{2}} - \left(b^{\frac{N}{2}}\right)^*$ is again a Hopf $*$-algebra.

On the other hand for $0 < q < 1$ the spectral condition is that spectra of a and b be contained in the closure of

$$\left\{ z \in \mathbb{C} : |z| \in q^{\mathbb{Z}} \right\} \subset \mathbb{C}.$$

This condition cannot be rephrased using only algebraic language.

The spectral conditions ensure the existence of C*-algebraic versions of the quantum "$az + b$" group for specified deformation parameters. They also ensure that the scaling group $(\tau_t)_{t\in\mathbb{R}}$ and unitary antipode R are unique. This last feature is typical for the C*-algebraic quantum groups (cf. [15]).

The quantum "$az + b$" groups described in this paper are constructed along the lines indicated in [17]. The corresponding spectral condition is of non algebraic nature, but many aspects of the construction are similar to the case of q being a root of unity.

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2 Special functions and commutation relations

2.1 Deformation parameter

The deformation parameter for our deformations of the "$az+b$" group will be a non zero complex number q such that if we write

$$q = \exp\left(\boldsymbol{\rho}^{-1}\right) \tag{2}$$

then $\operatorname{Re}\boldsymbol{\rho} < 0$ and

$$\operatorname{Im}\boldsymbol{\rho} = \frac{N}{2\pi},$$

where N is a non zero even integer. The set of such q with $|N| \leq 32$ is plotted in Figure 1.

The parameter $\boldsymbol{\rho}$ satisfying (2) will be kept fixed throughout the paper. Having chosen $\boldsymbol{\rho}$ we can compute complex powers of q:

$$q^z = \exp\left(\tfrac{z}{\boldsymbol{\rho}}\right)$$

for any $z \in \mathbb{C}$.

With q we associate the subgroup $\Gamma_q \subset \mathbb{C} \setminus \{0\}$ defined as the group generated by q and $\left\{q^{it} : t \in \mathbb{R}\right\}$.

Proposition 2.1 *The set Γ_q is a collection of $|N|$ logarithmic spirals around $0 \in \mathbb{C}$:*

$$\Gamma_q = \bigcup_{k=0}^{|N|-1} q^k \left\{q^{it} : t \in \mathbb{R}\right\}.$$

Consequently Γ_q is isomorphic to $\mathbb{Z}_{|N|} \times \mathbb{R}$.

It follows that Γ_q is a self dual locally compact abelian group. We shall denote by $\overline{\Gamma}_q$ the closure of Γ_q in $\mathbb{C}$, i.e. $\overline{\Gamma}_q = \Gamma_q \cup \{0\}$. Figure 2 presents Γ_q for $q = \exp\left(\boldsymbol{\rho}^{-1}\right)$ with $\boldsymbol{\rho} = -2 + i\frac{8}{2\pi}$.

2.2 Special functions

The self duality of Γ_q ensures existence of a nondegenerate bicharacter on Γ_q, but we shall need one with some additional properties:

Proposition 2.2 *There exists a unique continuous function $\chi\colon \Gamma_q \times \Gamma_q \to \mathbb{T}$ which is a symmetric and nondegenerate bicharacter, i.e.*

$$\begin{aligned} \chi(\gamma, \gamma') &= \chi(\gamma', \gamma), \\ \chi(\gamma, \gamma')\chi(\gamma, \gamma'') &= \chi(\gamma, \gamma'\gamma'') \end{aligned}$$

for all $\gamma, \gamma', \gamma'' \in \Gamma_q$ and

$$\left(\begin{array}{c} \chi(\gamma, \gamma') = 1 \\ \textit{for all } \gamma' \in \Gamma_q \end{array}\right) \Longrightarrow \Big(\gamma = 1\Big),$$

satisfying

$$\begin{aligned} \chi(\gamma, q) &= \operatorname{Phase}\gamma, \\ \chi(\gamma, q^{it}) &= |\gamma|^{it} \end{aligned}$$

for all $\gamma \in \Gamma_q$ and $t \in \mathbb{R}$. Moreover there exists a continuous function $\alpha\colon \Gamma_q \to \mathbb{T}$ such that for any $\gamma, \gamma' \in \Gamma_q$

$$\chi(\gamma, \gamma') = \frac{\alpha(\gamma\gamma')}{\alpha(\gamma)\alpha(\gamma')}.$$

The function α is obviously not unique. Although many aspects of our construction do not depend on the particular choice of α, we shall use

$$\alpha(q^n q^{it}) = \operatorname{Phase} \frac{(n+it)^2}{2\rho}.$$

Finally for $\gamma \in \Gamma_q \setminus \{-q^{-2k} : k \in \mathbb{Z}_+\}$ let

$$\mathbb{F}_q(\gamma) = \prod_{k=0}^{\infty} \frac{1 + \overline{q^{2k}\gamma}}{1 + q^{2k}\gamma}.$$

Then $\mathbb{F}_q$ extends to a continuous function $\overline{\Gamma}_q \to \mathbb{T}$. In particular $\mathbb{F}_q(0) = 1$. Some of the most important properties of $\mathbb{F}_q$ are summarized in the following theorem (cf. [8, Thm. 5.11, 5.16], [17, Thm. 1.1, 1.2],[13, Prop. 1.1, Eq. (1.3)]):

Theorem 2.3 **1.** *For any $\gamma \in \Gamma_q$ the function $\mathbb{R} \ni t \mapsto \mathbb{F}_q(q^{it}q\gamma)$ extends to a continuous function on the strip $\{z \in \mathbb{C} : -1 \leq \operatorname{Im} z \leq 0\}$ which is holomorphic in the interior of this strip. The value of the extension at the point $-i$ is $(1+\gamma)\mathbb{F}_q(\gamma)$.*

2. *For any $\gamma \in \Gamma_q$ we have*

$$\mathbb{F}_q(\gamma)\mathbb{F}_q(q^2\gamma^{-1}) = C_q \alpha(q^{-1}\gamma),$$

where C_q is a constant of absolute value 1.

The function $\mathbb{F}_q$ will be of utmost importance for the study of commutation relations in the next subsection. Point **1.** of Theorem 2.3 was in fact the property which led to the formula for $\mathbb{F}_q$. Point **2.** allows a detailed analysis of the asymptotic behaviour of $\mathbb{F}_q$.

2.3 Commutation relations

The commutation relations that need to be investigated in order to proceed with the construction of the our quantum "$az + b$" groups are the following:

$$\begin{aligned} SR &= q^2 RS, \\ SR^* &= R^* S, \end{aligned} \tag{3}$$

where (R, S) is a pair of normal operators acting on some Hilbert space. Of course, (3) is not a precise way of expressing relations between operators on a Hilbert space. It is obvious, however, that bounded non zero normal operators cannot satisfy (3).

It turns out that if one wants to consider relatively "simple" realizations (i.e. ones with phases and absolute values of R and S commuting up to a scalar, cf. [8, Sect. 6.1]) the correct definition is the following:

Definition 2.4 *Let H be a Hilbert space and let (R, S) be a pair of closed densely defined operators acting on H. We say that (R, S) is a q^2-pair if*

- *R and S are normal,*
- $\ker R = \{0\} = \ker S$,
- $\operatorname{Sp} R$, $\operatorname{Sp} S \subset \overline{\Gamma}_q$,
- *for all $\gamma, \gamma' \in \Gamma_q$ we have the* Weyl relation*:*

$$\chi(S, \gamma)\chi(\gamma', R) = \chi(\gamma', \gamma)\chi(\gamma', R)\chi(S, \gamma).$$

Theorem 2.5 *Let H be a Hilbert space and let (R, S) be a q^2-pair of operators acting on H. Then $S \circ R$, $R \circ S$, $S \circ R^*$ and $R^* \circ S$ are closable operators and their respective closures satisfy* (3)*.*

IDEA OF PROOF. Inserting in the Weyl relation (q, q^{it}), (q^{it}, q), (q, q) and $(q^{it}, q^{it'})$ in place of (γ, γ') and performing holomorphic continuation in the first two cases we get

$$\begin{aligned}
(\mathrm{Phase}\, S)\, |R| &= |q||R|\, (\mathrm{Phase}\, S)\,, \\
|S|\, (\mathrm{Phase}\, R) &= |q|\, (\mathrm{Phase}\, R)\, |S|, \\
(\mathrm{Phase}\, S)\, (\mathrm{Phase}\, R) &= (\mathrm{Phase}\, q)\, (\mathrm{Phase}\, R)\, (\mathrm{Phase}\, S)\,, \\
|S|^{it}|R|^{it'} &= |q^{it}|^{it'}|R|^{it'}|S|^{it}.
\end{aligned}$$

This set of equalities makes it possible to prove that there exists a common core for all finite products of operators from the set $\{S, S^*, R, R^*\}$. Then it is enough to check our relations on vectors from this common core.

It should be mentioned here that q^2-pairs exist and their structure is well known ([8, Prop. 6.9]). In fact any such pair is unitarily equivalent to a direct sum of so called *Schrödinger pairs.* The Schrödinger pair $(R_{\mathrm{S}}, S_{\mathrm{S}})$ acts on $L^2(\Gamma_q)$ in the following way: The operator R_{S} is just multiplication by the variable

$$\big(R_{\mathrm{S}} f\big)(\gamma) = \gamma f(\gamma),$$

while $\big(S_{\mathrm{S}} f\big)(\gamma)$ is the value of the analytic extension of $\mathbb{R} \ni t \mapsto f(q^{it} q\gamma)$ at the point $t = -i$.

Theorem 2.5 is true for more general pairs, but here we restrict our attention to pairs of operators which are distinguished (cf. [8, Sect. 6.2, 6.5]) by the following property:

Theorem 2.6 *Let H be a Hilbert space and let (R, S) be a q^2-pair of operators acting on H. Then*

1. *the sum $S + R$ is a closable operator and its closure $S \dotplus R$ satisfies*

$$S \dotplus R = \mathbb{F}_q(RS^{-1}) S \mathbb{F}_q(RS^{-1})^* = \mathbb{F}_q(R^{-1}S)^* R \mathbb{F}_q(R^{-1}S), \tag{4}$$

in particular $S \dotplus R$ is a normal operator and $\mathrm{Sp}\big(S \dotplus R\big) \subset \overline{\Gamma}_q$.

2. *The function $\mathbb{F}_q$ has the exponential property:*

$$\mathbb{F}_q(S \dotplus R) = \mathbb{F}_q(R) \mathbb{F}_q(S). \tag{5}$$

IDEA OF PROOF. First one has to prove that

$$S \dotplus RS = \mathbb{F}_q(R)^* S \mathbb{F}_q(R). \tag{6}$$

This follows from the property of the special function $\mathbb{F}_q$ described in point **1.** of Theorem 2.3. Assume that (R, S) is the Schrödinger pair. Then, since R is the operator of multiplication by the variable, we formally have $S\mathbb{F}_q(R) = (1 + R)\mathbb{F}_q(R)$ (cf. Theorem 2.3 and description of the Schrödinger pair).

In the second step one shows that if (R, S) is a q^2-pair then so is (RS^{-1}, S). Then applying (6) to this pair we get the second part of (4). The first part requires a few more tricks which include the use of point **2.** of Theorem 2.3.

To appreciate the power of this theorem more fully we need a result about the special function $\mathbb{F}_q$ and the affiliation relation.

Theorem 2.7 *Let H be a Hilbert space and let T be a normal operator acting on H such that* $\mathrm{Sp}\, T \subset \overline{\Gamma}_q$. *Let $A \subset B(H)$ be a nondegenerate* C^**-subalgebra. Then the following conditions are equivalent:*

1. *for all* $\gamma \in \overline{\Gamma}_q$ *the unitary operator* $\mathbb{F}_q(\gamma T)$ *belongs to* $M(A)$ *and the mapping*

$$\overline{\Gamma}_q \ni \gamma \longmapsto \mathbb{F}_q(\gamma T) \in M(A)$$

is strictly continuous,

2. *the operator* T *is affiliated with* A.

The key ingredient of the proof of this theorem the fact that the element $F \in M\big(C_\infty(\overline{\Gamma}_q) \otimes C_\infty(\overline{\Gamma}_q)\big)$ given by

$$F(\gamma, \gamma') = \mathbb{F}_q(\gamma\gamma')$$

($\gamma, \gamma' \in \overline{\Gamma}_q$) constitutes a quantum family of elements affiliated with $C_\infty(\overline{\Gamma}_q)$ which generates this algebra in the sense of [14, Sect. 4].

The algebraic consequences of commutation relations described in Definition 2.4 can now be revealed in the following theorem:

Theorem 2.8 *Let* H *be a Hilbert space,* $A \subset B(H)$ *a non degenerate* C**-subalgebra and let* (R, S) *be a* q^2*-pair such that* $R, S \,\eta\, A$. *Then*

1. *the operator* $S \dotplus R$ *is affiliated with* A,

2. *the operator* RS *is affiliated with* A.

Proof. **Ad 1.** By equation (5) we have for all $\gamma \in \overline{\Gamma}_q$

$$\mathbb{F}_q\big(\gamma(S \dotplus R)\big) = \mathbb{F}_q(\gamma R)\mathbb{F}_q(\gamma S)$$

and since composition of operators is a continuous operation (on bounded sets) on $M(A)$ Theorem 2.7 ensures that $(S \dotplus R) \,\eta\, A$.

Ad 2. Applying $\mathbb{F}_q$ to both sides of (6) we get

$$\mathbb{F}_q(S \dotplus RS) = \mathbb{F}_q(R)^* \mathbb{F}_q(S) \mathbb{F}_q(R).$$

On the other hand by (5) we have

$$\mathbb{F}_q(S \dotplus RS) = \mathbb{F}_q(RS)\mathbb{F}_q(S).$$

Combining these formulae we obtain

$$\mathbb{F}_q(RS) = \mathbb{F}_q(R)^* \mathbb{F}_q(S) \mathbb{F}_q(R) \mathbb{F}_q(S)^*. \tag{7}$$

Now reasoning presented in the proof of **1.** gives $RS \,\eta\, A$. Q.E.D.

Formula (7) appeared first in [7, Cor. 2.9]

3 The multiplicative unitary

Equipped with the tools of Section 2 we can proceed with the construction of the quantum group. We shall use the theory of manageable and, more generally, modular multiplicative unitaries ([15], [9], [1]). Before we define the operator from which our quantum group is built we will describe the "quantum space" of the group. This "quantum space" is the operator domain (cf. [17, Sect. 2], [2], [11]) denoted by G and defined as follows: for a Hilbert space H we shall write G_H for the set of pairs of normal operators (a, b) acting on H such that $\ker a = \{0\}$, a preserves $\ker b$ and $(b|_{\ker b^\perp}, a|_{\ker b^\perp})$ is a q^2-pair acting on $\ker b^\perp$. The correspondence $H \mapsto G_H$ is an operator domain which plays the role of the underlying space of our quantum group.

Now let us choose a Hilbert space H and a pair $(a, b) \in G_H$ such that $\ker b = \{0\}$. Then let

$$W = \mathbb{F}_q(b^{-1}a \otimes b)\chi(b^{-1} \otimes I, I \otimes a). \tag{8}$$

This will be our multiplicative unitary.

Proposition 3.1 *Let H be a Hilbert space and let $(a,b) \in G_H$ such that* $\ker b = \{0\}$. *Define W by* (8). *Then*

$$\begin{aligned} W(a \otimes I)W^* &= a \otimes a, \\ W(b \otimes I)W^* &= a \otimes b \dotplus I \otimes a. \end{aligned} \tag{9}$$

Proof. It is not difficult to show (cf. [8, Lem. 7.3]) that

$$\chi(b^{-1} \otimes I, I \otimes a)(a \otimes I)\chi(b^{-1} \otimes I, I \otimes a)^* = a \otimes a.$$

Then using the fact that (b,a) is a q^2-pair one can show that $a \otimes a$ strongly commutes with $b^{-1}a \otimes b$. It means that

$$\begin{aligned} W(a \otimes I)W^* &= \mathbb{F}_q(b^{-1}a \otimes b)\chi(b^{-1}a \otimes I, I \otimes a)(a \otimes I) \\ &\quad \times \chi(b^{-1} \otimes I, I \otimes a)^* \mathbb{F}_q(b^{-1}a \otimes b)^* \\ &= \mathbb{F}_q(b^{-1} \otimes b)(a \otimes a)\mathbb{F}_q(b^{-1}a \otimes b)^* \\ &= a \otimes a. \end{aligned}$$

For the proof of the second part of formula of (9) notice that the pair $(R,S) = (b \otimes I, a \otimes b)$ is a q^2-pair of operators acting on $H \otimes H$. Since $b \otimes I$ commutes with $\chi(b^{-1} \otimes I, I \otimes a)$, using the second formula (4) we obtain

$$\begin{aligned} W(b \otimes I)W^* &= \mathbb{F}_q(b^{-1}a \otimes b)\chi(b^{-1} \otimes I, I \otimes a)(b \otimes I) \\ &\quad \times \chi(b^{-1} \otimes I, I \otimes a)^* \mathbb{F}_q(b^{-1}a \otimes b)^* \\ &= \mathbb{F}_q(b^{-1}a \otimes b)(b \otimes I)\mathbb{F}_q(b^{-1}a \otimes b)^* \\ &= \mathbb{F}_q(R^{-1}S)R\mathbb{F}_q(R^{-1}S)^* = S \dotplus R \\ &= a \otimes b \dotplus b \otimes I. \end{aligned}$$

Q.E.D.

The operator W is a multiplicative unitary, i.e.

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

and the proof of this pentagon equation relies on the exponential formula (5).

In fact we have the following:

Proposition 3.2 *Let $(a,b) \in G_H$ be such that* $\ker b = \{0\}$ *and let $(\widehat{a}, \widehat{b}) \in G_K$, where K is some other Hilbert space. Define W by* (8) *and let*

$$V = \mathbb{F}_q(\widehat{b} \otimes b)\chi(\widehat{a} \otimes I, I \otimes a).$$

Then V is an operator adapted to W, i.e.

$$W_{23}V_{12} = V_{12}V_{13}W_{23}$$

on $K \otimes H \otimes H$.

The next result, which enables us to proceed with the construction of the quantum "$az+b$" group for new values of the deformation parameter, states that W is a modular multiplicative unitary ([9]):

Theorem 3.3 *Let H be a Hilbert space and let $(a,b) \in G_H$ be such that* $\ker b = \{0\}$. *Define W by* (8). *Then there exist positive selfadjoint operators Q and $\widehat{Q}$ on H such that* $\ker Q = \{0\} = \ker \widehat{Q}$ *and a unitary operator $\widetilde{W}$ on $\overline{H} \otimes H$ such that*

$$W\left(\widehat{Q} \otimes Q\right)W^* = \left(\widehat{Q} \otimes Q\right)$$

and for any $x,z\in H$, $u\in D(Q)$, $y\in D(Q^{-1})$

$$(x\otimes u|W|z\otimes y)=\left(\overline{z}\otimes Qu\middle|\widetilde{W}\middle|\overline{x}\otimes Q^{-1}y\right).$$

The operators Q, $\widehat{Q}$ and $\widetilde{W}$ are given by

$$\begin{gathered}Q=|a|,\qquad\qquad \widehat{Q}=|b|,\\ \widetilde{W}=\mathbb{F}_q\left(-(b^{-1}a)^\top\otimes b\right)^*\chi\left((b^{-1})^\top\otimes I,I\otimes a\right).\end{gathered}$$

Again the key to this result lies in the special functions introduced in Subsection 2.2. More precisely in the fact that the distributional Fourier transform of $\mathbb{F}_q$ on the locally compact abelian (and self dual) group Γ_q is

$$\widehat{\mathbb{F}_q}(\gamma)=-\frac{\mathbb{F}_q(-q^2)}{2\pi\overline{\boldsymbol{\rho}}}\frac{\chi(-q^{-2},\gamma)\gamma}{(1-\overline{\gamma})\mathbb{F}_q(-q^2\gamma)}.$$

4 The quantum group

4.1 The construction

The theory of manageable and modular multiplicative unitaries lets us now carry out the construction of our quantum group. Throughout this section we shall fix a Hilbert space H and a pair $(a,b)\in G_H$ such that $\ker b=\{0\}$.

The algebra playing the role of the algebra of continuous functions vanishing at infinity on our quantum group is by definition

$$A=\left\{(\omega\otimes\mathrm{id})W:\omega\in B(H)_*\right\}^{\text{norm closure}}. \tag{10}$$

It can be shown ([8, Prop. 7.25], cf. [17, Sect. 6]) that A is in fact isomorphic to a crossed product

$$A=C_\infty\left(\overline{\Gamma}_q\right)\rtimes_\beta\Gamma_q$$

where the action β of Γ_q on $C_\infty\left(\overline{\Gamma}_q\right)$ comes from the obvious action of Γ_q on $\overline{\Gamma}_q$ by multiplication.

Let us describe in more detail the relationship of operators a and b with the C*-algebra A. The inclusion $C_\infty\left(\overline{\Gamma}_q\right)\hookrightarrow M\left(C_\infty\left(\overline{\Gamma}_q\right)\rtimes_\beta\Gamma_q\right)$ is a morphism from $C_\infty\left(\overline{\Gamma}_q\right)$ to $C_\infty\left(\overline{\Gamma}_q\right)\rtimes_\beta\Gamma_q=A$. The operator b can be identified with the image under this morphism of the element $\boldsymbol{z}$ affiliated with $C_\infty\left(\overline{\Gamma}_q\right)$ given by

$$\boldsymbol{z}(\gamma)=\gamma$$

for all $\gamma\in\overline{\Gamma}_q$. It is therefore clear that $b\,\eta\,A$.

The operator a is the unique normal operator affiliated with A such that the unitary elements

$$\chi(a,\gamma)\in M(A)$$

($\gamma\in\Gamma_q$) constitute the unitary representation of Γ_q implementing the action β on the image of $C_\infty\left(\overline{\Gamma}_q\right)$ in $M(A)$. The existence of such an operator is a consequence of the famous SNAG theorem supplemented by a result similar to Theorem 2.7 ([8, Thm. 7.15]). Moreover it can be shown that a^{-1} is affiliated with A and the three elements

$$a,\ a^{-1},\ b$$

generate the C*-algebra A in the sense of [14, Sect. 3].

The theory of manageable multiplicative unitaries ensures the existence of a coassociative morphism $\delta \in \mathrm{Mor}(A, A \otimes A)$ given by

$$\delta(c) = W(c \otimes I)W^*$$

for any $c \in A$. It is now an easy consequence of Proposition 3.1 that the action of δ on affiliated elements a and b is

$$\begin{aligned} \delta(a) &= a \otimes a, \\ \delta(b) &= a \otimes b \dotplus b \otimes I. \end{aligned}$$

According to the general theory the scaling group $(\tau_t)_{t\in\mathbb{R}}$ of our quantum group is given by $\tau_t(c) = Q^{2it}cQ^{-2it}$ for all $c \in A$. Since $Q = |a|$ we see that τ_t is the unique automorphism of A such that

$$\begin{aligned} \tau_t(a) &= a, \\ \tau_t(b) &= q^{2it}b. \end{aligned} \tag{11}$$

It also follows from general theorems about manageable and modular multiplicative unitaries that the unitary antipode R of our quantum "$az + b$" group is given on generators by

$$\begin{aligned} a^R &= a^{-1}, \\ b^R &= -qa^{-1}b. \end{aligned} \tag{12}$$

Combining (11) and (12) we can compute the antipode $\kappa = R \circ \tau_{i/2}$:

$$\begin{aligned} \kappa(a) &= a^{-1}, \\ \kappa(b) &= -a^{-1}b. \end{aligned}$$

All these formulae agree with those found in the Hopf $*$-algebra framework (cf. Subsection 1.2). We shall denote our quantum group by $G = (A, \delta)$.

4.2 The dual group

The algebra interpreted as the algebra of continuous functions vanishing at infinity on the reduced dual of G is by definition

$$\widehat{A} = \left\{ (\mathrm{id} \otimes \omega)(W^*) : \omega \in B(H)_* \right\}^{\text{norm closure}}.$$

The comultiplication $\widehat{\delta}$ on $\widehat{A}$ is given by

$$\widehat{\delta}(d) = \sigma\big(W^*(I \otimes d)W\big),$$

(where σ is the flip automorphism of $B(H \otimes H)$) for all $d \in \widehat{A}$.

It will be convenient at this point to introduce the notation $\widehat{a} = b^{-1}$, $\widehat{b} = b^{-1}a$. The following theorem gives a complete description of the reduced dual $\widehat{G} = (\widehat{A}, \widehat{\delta})$ of G.

Theorem 4.1 *The operators $\widehat{a}$ and $\widehat{b}$ are affiliated with $\widehat{A}$ and there exists an isomorphism $\Psi \in \mathrm{Mor}\left(A, \widehat{A}\right)$ such that $\Psi(a) = \widehat{a}$, $\Psi(b) = \widehat{b}$ and*

$$\widehat{\delta}\big(\Psi(c)\big) = \sigma(\Psi \otimes \Psi)\delta(c).$$

for all $c \in A$.

The proof of this theorem is the same for quantum "$az+b$" groups for all values of the deformation parameter ([17, Sect. 7], [8, Sect. 7.5]).

The results of [5] can be easily transfered to the case where the deformation parameter assumes values described in Subsection 2.1. They say that for any Hilbert space K and any unitary element $V \in M(\mathcal{K}(K) \otimes A)$ such that

$$(\mathrm{id} \otimes \delta)V = V_{12}V_{13}$$

there exists a unique pair $(\widetilde{a}, \widetilde{b}) \in G_K$ such that

$$V = \mathbb{F}_q(\widetilde{b} \otimes b)\chi(\widetilde{a} \otimes I, I \otimes a).$$

This statement is converse to Proposition 3.2 and gives full description of strongly continuous unitary representations of G. In view of Theorem 4.1 this means that the quantum "$az+b$" group constructed in Subsection 4.1 is amenable.

4.3 The Haar measure and related topics

It was recently shown in [18] that the framework of modular multiplicative unitaries is very convenient for investigation of Haar weights. More precisely if W is a modular multiplicative unitary and $Q, \widehat{Q}$ and $\widetilde{W}$ are operators related to W as in Theorem 3.3 and A is the C*-algebra defined by (10) then

$$h(c) = \mathrm{Tr}\left(\widehat{Q}c\widehat{Q}\right)$$

defines a right invariant weight on A. If this weight is densely defined it is the right Haar measure of our quantum group.

It turns out that in the case of W defined by (8) the weight h is densely defined. As shown in [18, Example 3, Sect. 3] for an element

$$c = f(a)g(b) \in A$$

(where $f \in C_\infty(\Gamma_q)$, $g \in C_\infty\left(\overline{\Gamma}_q\right)$) we have

$$h(c^*c) = \int\limits_{\Gamma_q} |f(\gamma)|^2 \, d\mu(\gamma) \int\limits_{\Gamma_q} |g(\gamma)|^2 |\gamma|^2 \, d\mu(\gamma),$$

where $d\mu$ is a Haar measure on Γ_q. The scaling constant of this "$az+b$" group (i.e. the constant ν such that $h \circ \tau_t = \nu^{-t}h$) can now be easily computed: $\nu = \exp\left(4\,\mathrm{Im}\,\boldsymbol{\rho}^{-1}\right)$.

It should be remarked that the methods developed in [10] also work perfectly well for our deformation of the "$az+b$" group.

The quantum "$az+b$" group has a subgroup which is a classical group isomorphic to Γ_q. Using the theory of crossed products by abelian locally compact groups ([3]) one can obtain a satisfactory definition of the homogeneous space G/Γ_q. It turns out to be a classical space and it carries a natural action of G. These results will be collected in a separate paper [6].

Appendix

In the appendix we collected figures showing the allowed values of the deformation parameter q inside the unit circle in the complex plane and an example of the group Γ_q for $q = \exp\left(\boldsymbol{\rho}^{-1}\right)$ with $\boldsymbol{\rho} = -2 + i\frac{8}{2\pi}$.

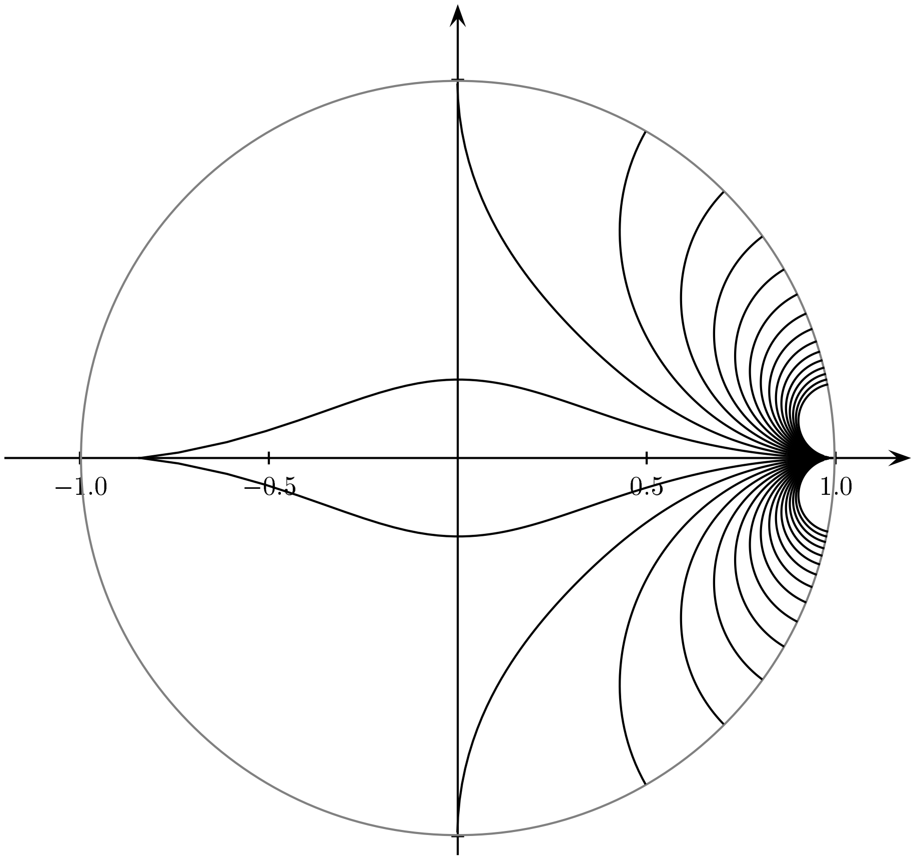

Figure 1: Values of the deformation parameter q ($N = \pm 2, \ldots, \pm 32$)

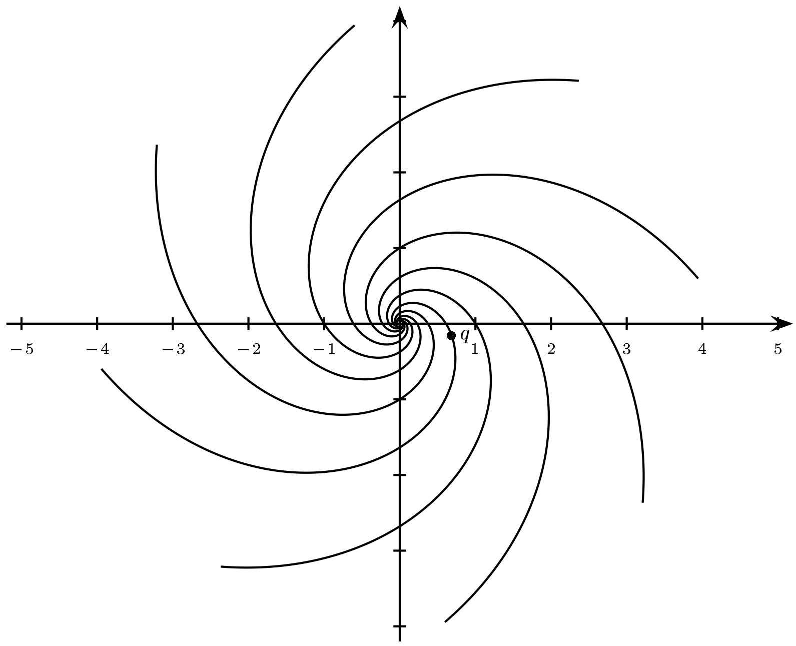

Figure 2: The group Γ_q for $\boldsymbol{\rho} = -2 + i\frac{8}{2\pi}$

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